$m=1$ and 2. The calculations are based upon the properties of an orthotropic material, namely 5 ply maple plywood. Frequency curves (frequency vs taper parameter) and transverse deflection curves for the first three modes of vibration with different boundary conditions are plotted in Figs. 1-4.

It is interesting to note that Ω increases with the increase in a/b, whereas Ω decreases with the increase in β. This is found true for all three modes of vibration and for all three edge conditions. Also Ω corresponding to β decreases slowly in the first and second modes but in the third mode it decreases rapidly. It is also found that Ω for the C-S-C-S plates is greater than the corresponding Ω for the C-S-S-S and S-S-S-S plates. This difference increases with the move toward the higher modes of vibration. For comparison purposes, Ω has been computed for an isotropic plate of parabolically varying thickness by converting the orthotropic parameters into the usual isotropic parameters. It compares well with the Ω of Ref. 1 under the identical conditions.

It is evident from Figs. 2-4 that the deflection of a plate of variable thickness is more than that for a plate of uniform thickness. This difference between the deflections increases with the increase in β. The nodal lines have also shifted toward the thinner side of the plate.

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Stability of Short Beck and Leipholz Columns on Elastic Foundation

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Introduction

STABILITY of nonconservative systems is extensively discussed in Ref. 1. It has been shown[2,3] that for slender uniform columns, resting on an elastic foundation of constant foundation modulus and subjected to nonconservative loads, the critical loads remain the same regardless of the value of the foundation modulus, and that the coalescence frequency† shifts by a quantity equal to the foundation modulus, compared to the column with no elastic foundation. The purpose of the present Note is to study the effect of shear deformation and rotatory inertia on a Beck column (a cantilever column with a concentrated follower force at the free end) and a Leipholz column (a cantilever column with a uniformly distributed follower force) on an elastic foundation and to examine whether the aforementioned phenomenon is applicable for short columns. The finite-element formulation presented in this Note for including the effects of shear deformation and rotatory inertia follows the same lines as Ref. 4 and the nonconservative stability problem is formulated using the standard formulation of Refs. 5 and 6.

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†The load on the column at which the two lowest frequencies (in the present study) become complex is the critical load and the corresponding frequency is the coalescence frequency.

Finite-Element Formulation

The matrix equation governing the present nonconservative stability problem is obtained as[5]

$$(\lambda^2+\Omega)[M]\{q\}-[K]\{q\}+Q([G^C]+[G^{NC}])\{q\}=0 \quad (1)$$

where $[K]$, $[M]$, $[G^C]$, $[G^{NC}]$, and $\{q\}$ are the assembled elastic stiffness matrix, mass matrix, geometric stiffness matrix for the conservative part of the load, geometric stiffness matrix for the nonconservative part of the load, and eigenvector, respectively. In Eq. (1), $\lambda^2=m\omega^2L^4/EI$, where m is the mass per unit length, ω the circular frequency, L the length of the column, E the Young's modulus, and I the moment of inertia, and $\Omega=kL^4/EI$, where k is the foundation modulus per unit length. For Beck's column, $Q=PL^2/\pi^2EI$ and for Leipholz's column, $Q=pL^3/\pi^2EI$, where P is the concentrated tip load and p the distributed load on the column per unit length.

The element stiffness matrix $[k]$, the mass matrix $[m]$, and the geometric stiffness matrices $[g^C]$ and $[g^{NC}]$ are obtained by using the standard procedure[7] from the expressions

$$U=\frac{1}{2}\int_0^l\left[EI\psi_x^2+5/6\left(\frac{EA}{2(1+\nu)}\epsilon_{xz}^2\right)\right]dx \quad (2)$$

Table 1 Critical loads Q_{cr} and coalescence frequencies λ_{cr}^2 for Beck's column for various L/r and Ω, eight-element solution

L/r	Ω	Q_{cr}	λ_{cr}^2
15	0.0	1.40	71.7
	0.1	1.40	71.8
	1.0	1.40	72.7
	10.0	1.39	81.6
	100.0	1.35	170.4
	1000.0	0.901	1048
25	0.0	1.74	97.6
	0.1	1.74	97.7
	1.0	1.74	98.6
	10.0	1.74	107.6
	100.0	1.72	196.9
	1000.0	1.51	1090
50	0.0	1.95	114.4
	0.1	1.95	114.5
	1.0	1.95	115.4
	10.0	1.95	124.4
	100.0	1.94	214.3
	1000.0	1.88	1112
100	0.0	2.01	119.5
	0.1	2.01	119.6
	1.0	2.01	120.5
	10.0	2.01	129.5
	100.0	2.01	219.5
	1000.0	1.99	1119
500	0.0	2.03	121.1
	0.1	2.03	121.2
	1.0	2.03	122.1
	10.0	2.03	131.1
	100.0	2.03	221.1
	1000.0	2.03	1121

Table 2 Critical loads Q_{cr} and coalescence frequencies λ^2_{cr} for various L/r and Ω, eight-element solution

L/r	Ω	Q_{cr}	λ^2_{cr}
15	0.0	2.64	70.9
	0.1	2.64	71.0
	1.0	2.64	71.9
	10.0	2.63	80.7
	100.0	2.55	169.0
	1000.0	1.85	1041
25	0.0	3.41	97.6
	0.1	3.41	97.7
	1.0	3.41	98.6
	10.0	3.40	107.5
	100.0	3.37	196.7
	1000.0	3.03	1087
50	0.0	3.87	114.7
	0.1	3.87	114.8
	1.0	3.87	115.7
	10.0	3.87	124.7
	100.0	3.86	214.4
	1000.0	3.76	1112
100	0.0	4.01	119.8
	0.1	4.01	119.9
	1.0	4.01	120.8
	10.0	4.01	129.8
	100.0	4.01	219.8
	1000.0	3.98	1119
500	0.0	4.07	121.2
	0.1	4.07	121.3
	1.0	4.07	122.2
	10.0	4.07	131.2
	100.0	4.07	221.2
	1000.0	4.07	1121

$$T=\frac{1}{2}\omega^2\int_0^l m[w^2+r^2(w_x+\gamma)^2]\mathrm{d}x \quad (3)$$

$$W^C=\frac{P}{2}\int_0^l w_x^2\mathrm{d}x, \quad \text{for Beck's column} \quad (4)$$

$$=\frac{p}{2}\int_0^l (l-x)w_x^2\mathrm{d}x, \quad \text{for Leipholz's column} \quad (5)$$

and

$$W^{NC}=-P(w_x+\gamma)\,|_{x=l}\,w(l), \quad \text{for Beck's column} \quad (6)$$

$$=-p\int_0^l (w_x+\gamma)w\mathrm{d}x, \quad \text{for Leipholz's column} \quad (7)$$

In the above expressions, l is the element length, ψ_x the curvature, A the area of cross section, and r the radius of gyration, w the lateral displacement, γ the shear rotation, and ν the Poisson ratio. Suffix x denotes differentiation with respect to the axial coordinate x.

For the derivation of the element matrices, w and γ are assumed as cubic in x with nodal parameters w, w_x, γ, and γ_x at each node.

Critical loads Q_{cr} are obtained with the use of the dynamic criterion where the first two frequencies coalesce.

Numerical Results

Using an eight-element idealization of the column, critical loads Q_{cr} and coalescence frequencies λ^2_{cr} are obtained for the Beck and Leipholz columns for various values of L/r and Ω. A convergence study is not made in the present Note as the element used has been proved to give very accurate results using an eight-element idealization.[6]

Table 1 gives the critical loads Q_{cr} and coalescence frequencies λ^2_{cr} of a Beck column for values of $L/r=15$, 25, 50, 100, and 500 and for values of $\Omega=0.0$, 0.1, 1.0, 10.0, 100.0, and 1000.0 and Table 2 gives the corresponding values of a Leipholz column.

The present results for both columns for $\Omega=0.0$ and for all values of L/r considered agree very well with those of Ref. 6. Also, for slender columns, i.e., for $L/r=500$, the present results confirm the phenomenon observed in Refs. 2 and 3 for both Beck and Leipholz columns.

Conclusions

From the results obtained in the present study, the following conclusions can be drawn:

1) For all the values of L/r considered in this Note, the critical loads Q_{cr} are same and λ^2_{cr} follows the shift phenomenon up to a value of $\Omega=10.0$.

2) For higher values of $\Omega>10.0$ and for lower values of $L/r<500$, the critical loads Q_{cr} decrease as Ω increases. This decrease is more pronounced in the case of low L/r; and for these parameters of Ω and L/r, λ^2_{cr} does not follow the shift phenomenon.

3) For slender columns, i.e., for $L/r=500$, the critical loads Q_{cr} are the same for any value of Ω and the coalescence frequency λ^2_{cr} follows the shift phenomenon.

4) The above conclusions are valid for both Beck and Leipholz columns.

5) The coalescence frequencies λ^2_{cr}, except for very low values of L/r, are the same for both Beck and Leipholz columns for a given value of Ω.

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